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ON STABILITY OF PLANE-PARALLEL CONVECTIVE MOTION DUE TO INTERNAL HEAT SOURCES

G. Z. GERSHUNI
State University, Perm, U.S.S.R.

E. M. ZHUKHOVITSKY
State Pedagogical Institute, Perm, U.S.S.R.

and

A. A. YAKIMOV
Computing Centre, State University, Perm, U.S.S.R.

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Abstract—The problem of the stability of steady-state convective motion in the plane inclined fluid layer with uniformly distributed heat sources is solved. The critical Grashof numbers and the critical disturbance parameters are determined as the function of the angle of inclination for some Prandtl numbers. The relationship between two-dimensional and three-dimensional disturbances is established. It is shown that depending on the Prandtl number and the angle of inclination, the crisis of steady motion is either of hydrodynamic nature or is caused by the Rayleigh instability of the stratified fluid.

NOMENCLATURE

x, y, z,	Cartesian co-ordinates;
h,	layer thickness;
g,	acceleration of gravity;
ρ,	mean density;
$\mathbf{v}(v_x, v_y, v_z)$,	velocity;
p,	pressure;
p,	pressure;
T,	temperature, temperature disturbance;
Q,	volumetric density of internal heat sources;
$q = Q/\rho c_p \chi$,	reduced density of heat sources;
v_0, T_0, p_0,	velocity, temperature and pressure in the steady regime;
$G = g\beta q h^5/2\nu^2$,	Grashof number;
$P = \nu/\chi$,	Prandtl number;
$R = G.P$,	Rayleigh number;
$k, \tilde{k}$,	wave-number of two-dimensional disturbance;
k_y, k_z,	wave-numbers of three-dimensional disturbance;
a_i, b_k,	expansion coefficients in Galerkin's method;
$H_{ni}, B_{km}, E_{mi}, D_{kn}, C_{kn}$,	matrix elements;
G_m,	minimum critical Grashof number;
k_m, c_m,	wave-number and phase velocity of critical disturbance;
$a = k_z/(k_y^2 + k_z^2)^{1/2}$,	three-dimensional disturbance parameter;
α,	angle of inclination of the layer to the vertical;
β, ν, χ,	coefficients of thermal expansion, kinematic viscosity and thermal diffusivity;
γ,	unit vector along the upward vertical;
ψ,	streamfunction of two-dimensional disturbances;
$\lambda = \lambda_r + i\lambda_i$,	decrement of normal disturbance;
$\varphi(x), \theta(x)$,	disturbance amplitudes of streamfunction and temperature;
φ_i, θ_k,	base functions;
μ_i, ν_k,	disturbance decrements at $G = 0$.

INTRODUCTION

IN A NUMBER of works [1–9] the stability of the steady plane-parallel convective motion in the fluid layer between the heated parallel planes at different temperatures is studied in detail. The solution of the boundary-value problem for the amplitudes of small normal disturbances has allowed the spectrum of the characteristic disturbances to be determined and the boundary of the flow stability to be found. Convective motion with the odd (cubic) velocity profile was found to exhibit the instability of two types at a rather large temperature difference. At small and moderate Prandtl numbers, in the case of the vertical layer orientation, the instability is of hydrodynamic nature and is due to formation of steady eddies at the interface of convective counterflows. At rather large Prandtl numbers there appears and becomes most dangerous a new instability mode due to development of disturbances of the type of the amplified travelling heat waves in the flow. The situation changes essentially in the case of inclined orientation of the layer. If the layer is inclined to the vertical so that the lower plane has a higher temperature, then there also appears the Rayleigh-type instability due to the density stratification in the fluid heated from below. For the inclined orientation of the layer the above mechanisms of the instability interact, that, generally speaking, leads to a rather complex situation. In particular, the relationship between the two-dimensional and three-dimensional disturbances is considerably complicated. So, if for the vertical orientation the two-dimensional disturbances are most dangerous, then in the Rayleigh region of the inclination angles the crisis is due to three-dimensional disturbances (at $P > 0{\cdot}25$).

Recently the study has been made of one more interesting type of convective motion due to internal heat sources. In [10–11] the vertical orientation of the layer with uniformly distributed heat generation within the volume has been studied. In this case, unlike the flow between the heated planes at different temperatures, the steady flow is characterized by the even velocity and temperature profiles, that results in essential peculiarities of the structure of disturbance spectra and the instability form. In [10] the problem has been solved with a pure hydrodynamic statement: the effect of the thermal factors on the disturbance development is neglected. To this approximation, which corresponds to the limiting case of small Prandtl numbers, the stability boundary has been found, and the basic level of the instability is shown to be related to the development of two eddy systems at the interface of the upward and downward streams. The solution of the problem with a complete statement [9, 11] has shown that the effect of thermal factors on the stability is very essential. With increasing Prandtl numbers the Grashof number G_m defining the boundary of the steady-state flow stability decreases greatly, and at $P \to \infty$ the asymptotic law $G_m \sim P^{-1/2}$ holds. The phase velocity of the neutral disturbances increases with P, and the disturbances themselves become of the travelling thermal wave type.

The aim of the present work is to study the flow stability with internal heat generation at the arbitrary orientation of the layer. To solve the amplitude boundary-value problem, Galerkin's method is used. The disturbance spectra and critical Grashof numbers are found as the function of the problem parameters for different orientations of the layer. It is shown that there exists a transformation which allows all information on three-dimensional disturbances to be obtained by solving the boundary-value problem on two-dimensional disturbances (analog of the Squire transformation). With the help of this transformation the stability boundary is found for three-dimensional disturbances, and these disturbances are shown to be more dangerous over a wide range of the parameters (inclination angle and Prandtl number).

STEADY MOTION

Consider the plane fluid layer $2h$ thick between parallel plates. The layer is inclined to the vertical at the angle α (Fig. 1). The boundary planes are kept at the same constant temperature. The internal heat sources with the strength Q are uniformly distributed within the fluid volume. The layer length is rather large, and the steady motion in the region of the layer far from the ends may therefore be considered plane-parallel.

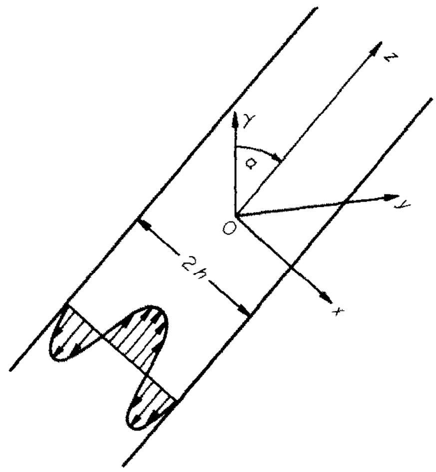

FIG. 1. Coordinate axes and velocity profile of the main motion.

The heat convection equations in the Boussinesq approximation is written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\boldsymbol{v} + g\beta T\boldsymbol{\gamma}, \qquad (1)$$

$$\frac{\partial T}{\partial t} + \boldsymbol{v}\nabla T = \chi\Delta T + \frac{Q}{\rho c_p}, \qquad (2)$$

$$\operatorname{div} \boldsymbol{v} = 0. \qquad (3)$$

For the steady-state plane-parallel motion we have (the co-ordinates are shown in Fig. 1)

$$v_x = v_y = 0, \; v_z = v_0(x), \; T = T_0(x), \; p = p_0(z). \qquad (4)$$

Steady distributions of the velocity $v_0(x)$, temperature $T_0(x)$, and pressure $p_0(z)$ are found from the equations

$$\nu \frac{d^2 v_0}{dx^2} + g\beta T_0 \cos\alpha = \frac{1}{\rho}\frac{dp_0}{dz} = C, \quad (5)$$

$$\frac{d^2 T_0}{dx^2} = -q, \quad (q = Q/\rho c_p \chi), \quad (6)$$

where C is the constant of variable separation defining the longitudinal pressure gradient.

The boundary conditions will be formulated as follows. On both planes $x = \pm h$ the velocity vanishes, and the temperature has a constant value taken as the reference point

$$x = \pm h: v_0 = 0,\ T_0 = 0. \quad (7)$$

The remote ends of the channel are assumed to be closed. This leads to the condition of a vanishing flow rate through the layer cross-section

$$\int_{-h}^{h} v_0(x)\, dx = 0. \quad (8)$$

The solution of equations (5), (6) under conditions (7) and (8) is of the form:

$$v_0(x) = \frac{g\beta q h^4 \cos\alpha}{120\nu}\left[1 - 6\left(\frac{x}{h}\right)^2 + 5\left(\frac{x}{h}\right)^4\right], \quad (9)$$

$$T_0(x) = \frac{qh^2}{2}\left[1 - \left(\frac{x}{h}\right)^2\right], \quad (10)$$

$$p_0(z) = \tfrac{2}{5}\rho g\beta q h^2 \cos\alpha \,.\, z + \text{const}. \quad (11)$$

Thus, in the steady-state plane-parallel flow the temperature distribution is quadratic over the cross-section, and the velocity profile describes the central upward and two downward convective flows (this profile is shown in Fig. 1). From (9) it may be seen that the velocity of motion is maximum for the vertical orientation of a layer ($\alpha = 0$), and at $\alpha \to \pi/2$ the velocity tends to zero (equilibrium).

STATEMENT OF THE PROBLEM ON STABILITY

If the strength of internal sources is sufficiently large, then the convective motion (9)–(11) becomes very intensive and may therefore appear to be unstable. To analyze the stability, the disturbed motion $v_0 + v$, $T_0 + T$, $p_0 + p$ will be considered where v, T, p are small disturbances depending on time. The equations for small disturbances are obtained from (1)–(3) by their linearization near the steady-state solution (9)–(11). Equations for small disturbances are written in a dimensionless form with the following units: h is the distance; h^2/ν is the time; $g\beta q h^4/2\nu$ is the velocity; $qh^2/2$ is the temperature; $\rho g\beta q h^3/2$ is the pressure. The dimensionless disturbance equations are of the form:

$$\frac{\partial \boldsymbol{v}}{\partial t} + G[(\boldsymbol{v}\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\nabla)\boldsymbol{v}] = -\nabla p + \Delta\boldsymbol{v} + T\boldsymbol{\gamma}, \quad (12)$$

$$\frac{\partial T}{\partial t} + G[(\boldsymbol{v}\nabla T_0) + (\boldsymbol{v}_0\nabla T)] = \frac{1}{P}\Delta T. \quad (13)$$

$$\operatorname{div} \boldsymbol{v} = 0 \quad (14)$$

where

$$G = \frac{g\beta q h^5}{2\nu^2} \quad \text{and} \quad P = \frac{\nu}{\chi}$$

are the Grashof and Prandtl numbers; v_0 and T_0 are the dimensionless velocity and temperature profiles of the main flow

$$v_0(x) = \frac{\cos\alpha}{60}(1 - 6x^2 + 5x^4) \equiv \cos\alpha f_0(x), \quad (15)$$

$$T_0(x) = 1 - x^2. \quad (16)$$

At the layer boundaries the disturbances of a velocity and temperature vanish

$$x = \pm 1: v = 0,\ T = 0. \quad (17)$$

It will be further seen that when studying the stability of convective motion the two-dimensional disturbances play an important role. Assuming

$$v_x = -\frac{\partial\psi}{\partial z}, \quad v_y = 0, \quad v_z = \frac{\partial\psi}{\partial x} \quad (18)$$

(ψ is the streamfunction) and eliminating the pressure, the equations for two-dimensional disturbances are obtained:

$$\frac{\partial}{\partial t}\Delta\psi + G\cos\alpha\left(f_0\frac{\partial}{\partial z}\Delta\psi - f_0''\frac{\partial\psi}{\partial z}\right) = \Delta^2\psi + \left(\cos\alpha\frac{\partial T}{\partial x} + \sin\alpha\frac{\partial T}{\partial z}\right), \quad (19)$$

$$\frac{\partial T}{\partial t} + G\left(f_0\cos\alpha\frac{\partial T}{\partial z} - T_0'\frac{\partial\psi}{\partial z}\right) = \frac{1}{P}\Delta T. \quad (20)$$

Introduce the normal disturbances

$$\psi = \varphi(x)\exp[-\lambda t + ikz]$$
$$T = \theta(x)\exp[-\lambda t + ikz]. \quad (21)$$

Here φ and θ are the amplitudes, k is the real wave number for the periodicity of disturbances along the z-axis; λ is the disturbance decrement. Substitution of (21) into (19) and (20) gives the amplitude equations

$$\Delta^2\varphi - ikG\cos\alpha \,.\, H\varphi + \cos\alpha\theta' + ik\sin\alpha\theta = -\lambda\Delta\varphi, \quad (22)$$

$$\frac{1}{P}\Delta\theta + ikG(T_0'\varphi - \cos\alpha f_0\theta) = -\lambda\theta. \quad (23)$$

Here the operator notations are introduced

$$\Delta = \frac{d^2}{dx^2} - k^2, \quad H = f_0\Delta - f_0''.$$

The amplitudes φ and θ satisfy the homogeneous boundary conditions

$$x = \pm 1: \varphi = \varphi' = 0, \ \theta = 0. \quad (24)$$

The amplitude boundary-value problem (22)–(24) determines the spectrum of characteristic disturbances and their decrements $\lambda = \lambda_r + i\lambda_i$. The sign of the real part of λ_r defines damping ($\lambda_r > 0$) or growth ($\lambda_r < 0$) of disturbances; the stability boundary is found from the condition $\lambda_r = 0$. The imaginary part of the decrement λ_i gives the frequency of the disturbance oscillations and their phase velocity.

METHOD OF SOLUTION

For approximate solution of the spectral problem (22)–(24) Galerkin's method is used. The disturbance amplitudes of the streamfunction and temperature are presented in a series form with respect to some systems of base functions

$$\begin{aligned} \varphi &= a_0\varphi_0 + a_1\varphi_1 + \ldots + a_{N-1}\varphi_{N-1}, \\ \theta &= b_0\theta_0 + b_1\theta_1 + \ldots + b_{M-1}\theta_{M-1}. \end{aligned} \quad (25)$$

Eigenfunctions of the following boundary-value problem will be chosen as the base functions of φ_i

$$\Delta^2\varphi_i = -\mu_i\Delta\varphi_i, \quad \varphi_i(\pm 1) = \varphi_i'(\pm 1) = 0. \quad (26)$$

The base functions θ_k are determined by the boundary-value problem

$$\frac{1}{P}\Delta\theta_k = -\nu_k\theta_k, \quad \theta_k(\pm 1) = 0. \quad (27)$$

The significance of the functions φ_i and θ_k is quite clear: these are the amplitudes of velocity and temperature disturbances in the fluid layer at rest with no heating ($G = 0$)*.

Substituting series (25) into amplitude equations (22) and (23) and making up the orthogonality conditions of Galerkin's method yield the homogeneous linear algebraic system of $N+M$ equations for the coefficients a_i, b_k:

$$\begin{aligned} &\sum_{i=0}^{N-1} [(\mu_i - \lambda)\delta_{in} - ikG\cos\alpha . H_{in}]a_i \\ &\quad + \sum_{k=0}^{M-1} [\cos\alpha . D_{kn} + ik\sin\alpha . C_{kn}]b_k = 0 \\ &\qquad\qquad (n = 0, 1, \ldots, N-1) \\ &\sum_{k=0}^{M-1} [(\nu_k - \lambda)\delta_{km} + ikG.\cos\alpha . B_{km}].b_k \\ &\quad - ikG\sum_{i=0}^{N-1} E_{mi}a_i = 0 \\ &\qquad\qquad (m = 0, 1, \ldots, M-1) \end{aligned} \quad (28)$$

*The base of φ_i was suggested and used to solve the Orr–Sommerfeld problem by Petrov [12, 13]. It was used for studying the disturbance spectrum of isothermal flows [14–16]. The complete base of φ_i, θ_k was used to study the disturbance spectrum and the stability of the convective flows between heated planes at different temperatures [3, 4, 8, 9].

where δ_{ik} is the Kronecker delta, and the notations of the matrix elements are introduced

$$H_{ni} = \int_{-1}^{1} \varphi_n H\varphi_i\,dx, \qquad B_{km} = \int_{-1}^{1} \theta_k f_0 \theta_m\,dx,$$

$$E_{mi} = \int_{-1}^{1} T_0'\varphi_i\theta_m\,dx, \qquad D_{kn} = \int_{-1}^{1} \theta_k'\varphi_n\,dx,$$

$$C_{kn} = \int_{-1}^{1} \theta_k\varphi_n\,dx.$$

The normalized functions φ_i and θ_k are assumed

$$\int_{-1}^{1} \varphi_i\Delta\varphi_i\,dx = -1, \qquad \int_{-1}^{1} \theta_k^2\,dx = 1.$$

The condition for existence of a non-trivial solution of system (28) leads to a dispersion equation, from which the characteristic decrements of the disturbances λ are found as the function of the parameters k, G, P, α. To determine the spectrum λ, it is thus necessary to find the eigenvalues of the complex matrix corresponding to system (28).

Diagonalization of the matrix was performed numerically on the electronic computer with the help of the QR-algorithm [17]. In the calculations the use was made of approximations (25) which contain 8–14 functions in the φ and θ series. The convergence was checked by comparing the approximations with the different number of base functions. The comparison has shown that the approximations used give the sufficient accuracy of the decrements and the critical Grashof numbers over the range of the parameters considered

DISCUSSION OF RESULTS. PLANE DISTURBANCES

Now the numerical results obtained will be discussed. First the vertical orientation of the layer ($\alpha = 0$) will be briefly considered. Figure 2 gives the neutral curves of Grashof number vs wave number for three values of P. With an increase in P, the minimum critical number G_m decreases monotonically. A plot of G_m vs P is presented in Fig. 3. At large P the asymptotic relation

$$G_m = 488/\sqrt{P} \quad (29)$$

is valid.

The comparison of the neutral curves for different values of P shows strong thermal effects on the stability. Even at $P = 1$ the values of G_m are considerably lower, as compared to the limiting value $G_m = 1720$, corresponding to purely hydrodynamic approximation ($P = 0$). As P increases the minimum on the neutral curve is shifted towards the small k, i.e. it passes to the disturbances with a larger wave-length. It is interesting that as far as P increases, the neutral curve shape

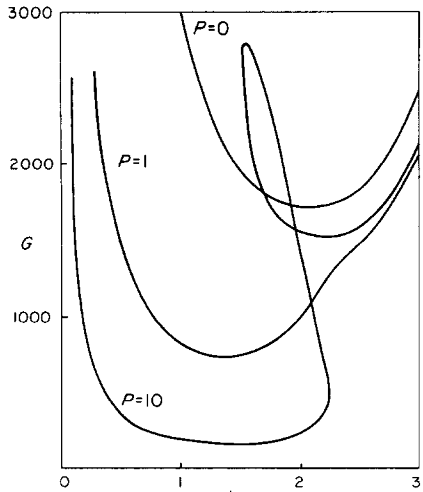

FIG. 2. Neutral curves (vertical layer).

where λ_i is the imaginary part of the dimensionless decrement, and k_m and G_m are the parameters of the minimum point on the neutral curve. At all P the phase velocity C_m is negative (critical disturbances are the travelling waves which propagate downward). At $P = 0$ the drift speed of the critical disturbances is small ($C_m = -0{\cdot}16$). As far as P increases the phase velocity grows monotonically, and at $P = 20$, for example, it exceeds the maximum velocity of the undisturbed flow: $C_m = -1{\cdot}36$.

Now we shall discuss the results on the inclined layer. Calculations were made for the Prandtl numbers $P = 0{\cdot}1$, 1 and 10.

Figures 4(a–c) gives families of the neutral curves for various slopes and the above values of the Prandtl number. It may be seen that at $P = 0{\cdot}1$ for all angles of inclinations the crisis occurs at the same instability mode. At $P = 1$ and $P = 10$ the structure of the

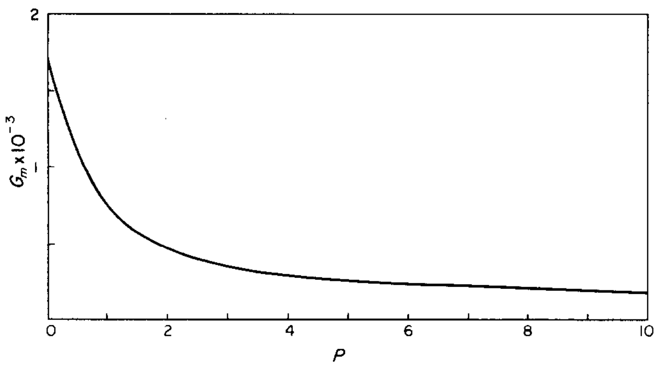

FIG. 3. Critical Grashof number vs Prandtl number for a vertical layer.

changes essentially, and at $P > 5{\cdot}7$ a closed loop* appears on the curve. Thus, at large P the neutral curve essentially consists of two branches, the absolute minimum determining the crisis of steady motion being achieved on the long-wave branch.

As far as the Prandtl number increases, the mode responsible for instability is replaced: while at small P instability is caused by hydrodynamic disturbances (μ—levels of the spectrum), then with an increase in P the instability passes to the disturbances of the thermal wave type (ν—levels). The replacement of the instability mode is accompanied by a considerable growth of the phase velocity of critical disturbances. This velocity in the units of the maximum velocity of the upward flow on the channel axis $g\beta q h^4/120\,\nu$ may be written as:

$$C_m = \frac{60\,\lambda_i}{k_m G_m} \tag{30}$$

*A similar peculiarity in the shape of neutral curves is found when studying the stability of a convective boundary layer near a heated vertical plate [18].

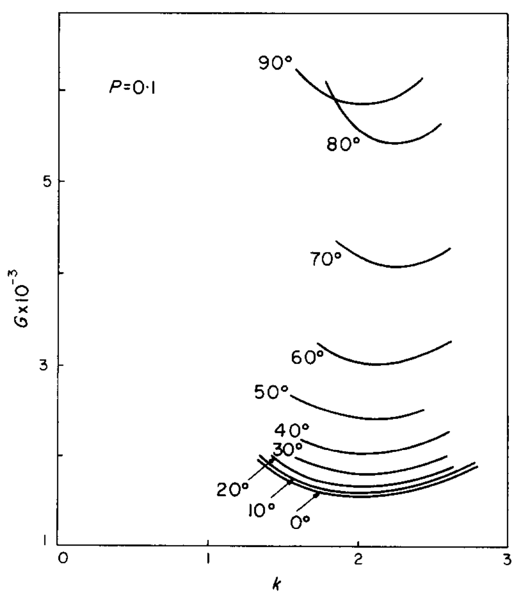

FIG. 4(a). Neutral curves for different angles of inclination: $P = 0{\cdot}1$.

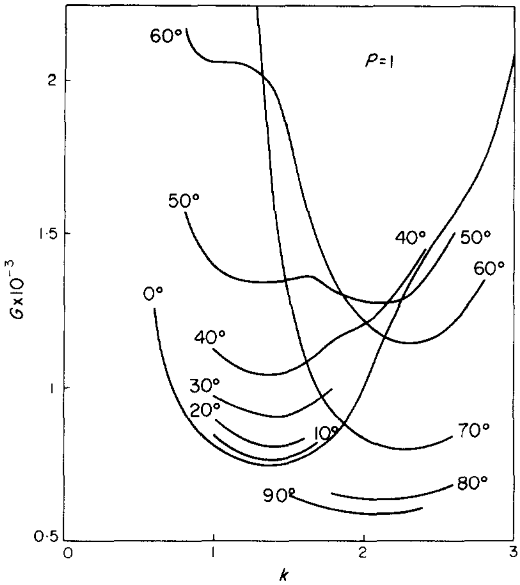

Fig. 4(b). Neutral curves for different angles of inclination: $P = 1$.

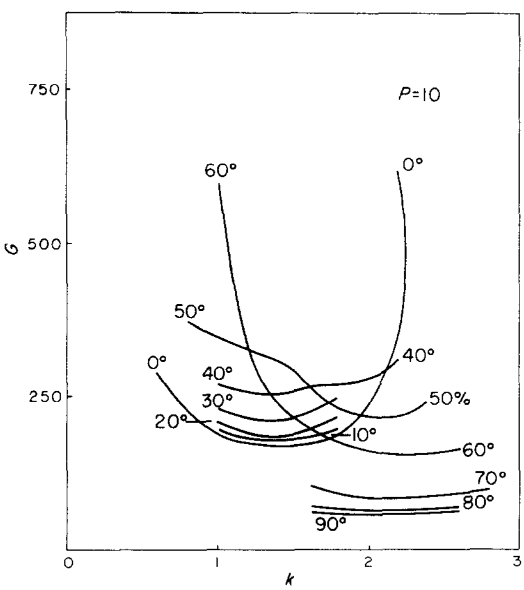

Fig. 4(c). Neutral curves for different angles of inclination: $P = 10$.

families of the neutral curves is more complicated due to the presence of two minima (over a certain range of angles). As far as the angle of inclination to the vertical α increases, the critical number G_m firstly grows, and then when the absolute minimum passes to another branch of the neutral curve G_m start decreasing. The replacement of the instability mode occurs at $P = 1$ and $P = 10$ at the angles $\alpha = 49°$ and $\alpha = 41°$, respectively. Owing to the fact that the absolute minimum passes to another branch of the neutral curve the critical wave number k_m (Fig. 5) and the phase velocity of critical disturbances C_m (Fig. 6) as the function of the inclination angle undergo a jump.

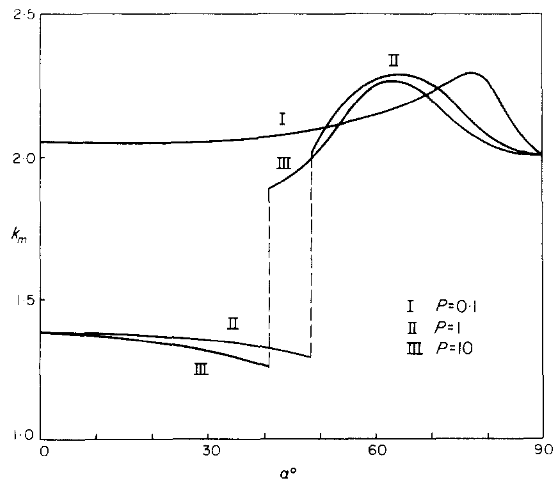

Fig. 5. Critical wave number vs the angle of inclination.

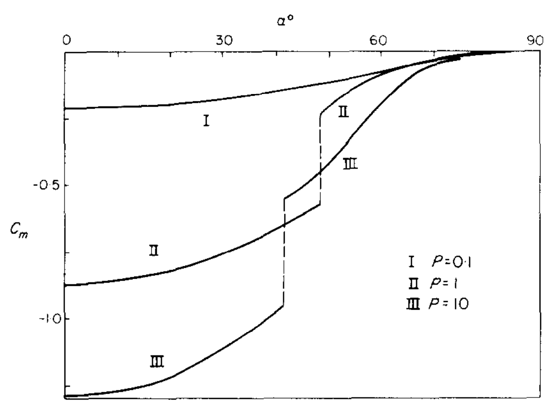

Fig. 6. Phase velocity of critical disturbance vs the angle of inclination.

The stability boundary vs α for three above values of P is shown in Fig. 7 in the co-ordinates $G_m P$, α. As is seen, at $\alpha = 90°$ the product $G_m . P = R_m$ (critical Rayleigh number) does not depend on P. This may be expected: the limiting case $\alpha = 90°$ corresponds to the horizontal layer of the fluid at rest stratified by the internal heat sources along the vertical. In this case the problem is to study the fluid equilibrium stability with the parabolic temperature distribution along the vertical. As is known, the equilibrium crisis of the heated fluid is determined by the Rayleigh number. At $\alpha = 90°$

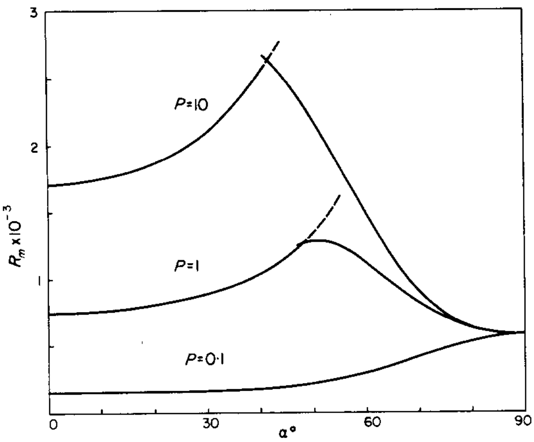

FIG. 7. Minimum critical Rayleigh number vs the angle of inclination.

the critical Rayleigh number appears to be equal to $R_m = 584$, and the critical wave number, $k_m = 2{\cdot}00$. These parameters agree with the values obtained when studying the stability of the horizontal layer equilibrium in [19].

As is seen from Fig. 6, at $\alpha \to 90°$ the phase velocity tends to zero. This also corresponds to the equilibrium instability where, as is known, the crisis is related to steady disturbances (fixed convective cells).

THREE-DIMENSIONAL DISTURBANCES

The results presented in the previous section refer to the two-dimensional disturbances, whose all variables are independent of y and $v_y = 0$. To study the behaviour of three-dimensional disturbances in the steady motion (9)–(11), it is necessary to refuse from assumptions (18), (21) and to consider more general disturbances

$$(v_x, v_y, v_z, T, p) \sim \exp[-\lambda t + i(k_y y + k_z z)], \quad (31)$$

where now $v_y \neq 0$, and k_y and k_z are the wave numbers for periodicity along the axes y and z. Substitution of (31) into general disturbance equations (12)–(14) gives the system of the equations for amplitudes of three-dimensional disturbances as a function of x. This system differs from the appropriate one for disturbances in the flow between heated planes at different temperatures only by the type of velocity and temperature profiles of steady motion. It is therefore possible to use the results of [5]. These are as follows. The boundary-value problem for three-dimensional disturbances may be reduced to that for two-dimensional ones with the help of some transformations of the unknown functions and parameters of the problem (analog of the known Squire transformations). Thus, all information on stability regarding three-dimensional disturbances may be obtained from the above solution of the boundary-value problem for two-dimensional disturbances. As is shown in [5], the decrement λ, the critical Grashof number for three-dimensional disturbances with the wave numbers k_y and k_z for the layer oriented at the angle α to the vertical may be found if the decrement $\tilde{\lambda}$, critical Grashof number $\tilde{G}$, for two-dimensional disturbances with the wave number $\tilde{k}$ for the layer inclined to the vertical at the angle $\tilde{\alpha}$ different from α are known. The transformations of two-dimensional quantities marked by the sign "~" into three-dimensional ones are as follows:

$$\lambda = \tilde{\lambda}, \qquad k_y^2 + k_z^2 = \tilde{k}^2, \qquad \tan\alpha = \frac{k_z}{\tilde{k}}\tan\tilde{\alpha}$$
$$G = \tilde{G}\sqrt{\sin^2\tilde{\alpha} + \left(\frac{\tilde{k}}{k_z}\right)^2 \cos^2\tilde{\alpha}}. \quad (32)$$

If the layer is vertical ($\alpha = 0$), then from (32) we arrive at:

$$\tilde{\alpha} = 0, \qquad G = \tilde{G}\frac{\tilde{k}}{k_z}. \quad (33)$$

Since the parameter $k_z/\tilde{k} = k_z/\sqrt{k_y^2 + k_z^2} \leqslant 1$, hence it is seen that $G \geqslant \tilde{G}$, i.e. for the vertical orientation higher critical Grashof numbers correspond to three-dimensional disturbances. As in the case of isothermal flows, two-dimensional disturbances with $k_y = 0$ are most dangerous.

For the horizontal layer ($\alpha = 90°$) we have

$$\tilde{\alpha} = 90°, \qquad G = \tilde{G} \quad (34)$$

i.e. the critical numbers for two- and three-dimensional disturbances coincide.

Now consider the range of the angles $0° < \alpha < 90°$. To describe three-dimensional disturbances, it is convenient to introduce the parameter $a = k_z/\sqrt{k_y^2 + k_z^2}$. The values of this parameter are written within the range $0 \leqslant a \leqslant 1$. The limiting case $a = 1 (k_y = 0)$ corresponds to two-dimensional disturbances; the case $a = 0 (k_z = 0)$ corresponds to three-dimensional disturbances shaped as rolls, whose axes are parallel to the velocity of the main motion ("z-rolls"). Intermediate values of "a" describe three-dimensional disturbances with an arbitrary ratio of the wave numbers k_z/k_y. The critical parameters of three-dimensional disturbances with a fixed value of "a" may be obtained from transformations (32).

Figures 8(a–c) gives the families of the curves for $G_m(\alpha)$ at different "a" with $P = 0{\cdot}1$, 1 and 10. As is seen from Fig. 8(a), at small Prandtl numbers ($P = 0{\cdot}1$) the plane disturbances are most dangerous throughout the whole range of angles of inclination ($a = 1$). Quite a different situation occurs at not small P. As is seen from Fig. 8(b, c) at $P = 1$ and $P = 10$ the plane disturbances are most dangerous if the angle of inclination

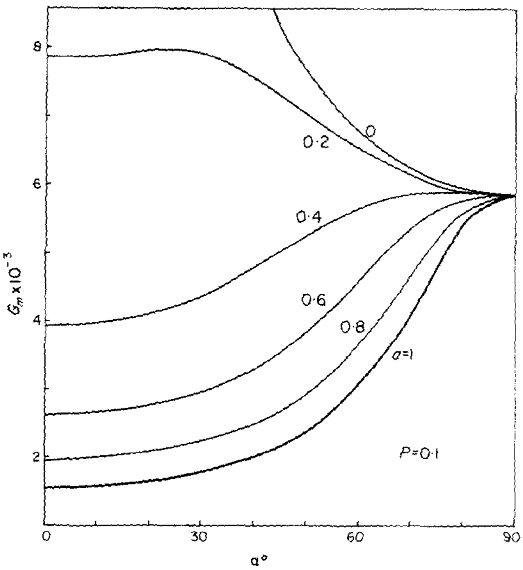

FIG. 8(a). Critical Grashof numbers of three-dimensional disturbances vs the angle of inclination: $P = 0{\cdot}1$.

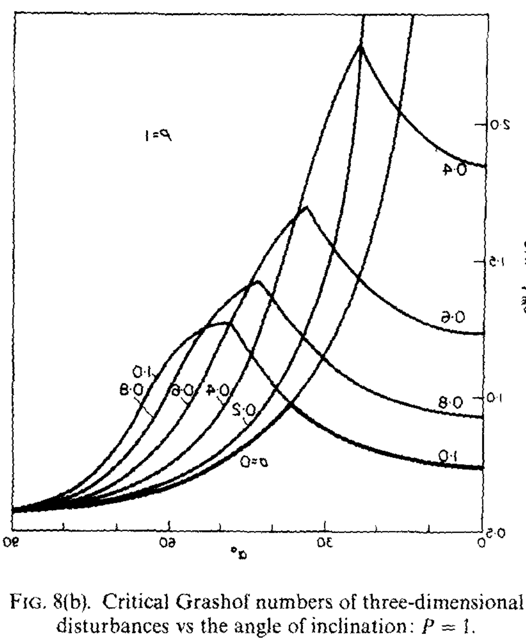

FIG. 8(b). Critical Grashof numbers of three-dimensional disturbances vs the angle of inclination: $P = 1$.

to the vertical is less than some α_*. At $\alpha > \alpha_*$ the absolute minimum of the critical number G_m passes to the three-dimensional disturbances of the "z-roll" type ($a = 0$). With increasing P the angle α_* decreases: at $P = 1$ and $P = 10$ $\alpha_* = 37°$ and $\alpha_* = 18°$, respectively. At large P the "z-roll" type disturbances are thus responsible for the crisis of the steady motion over a

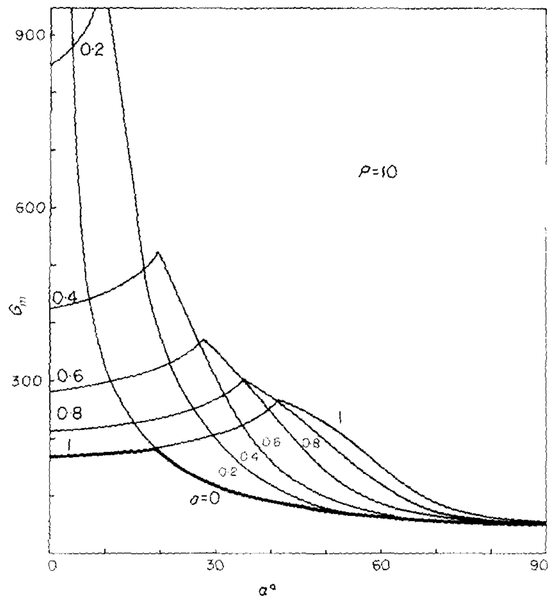

FIG. 8(c). Critical Grashof numbers of three-dimensional disturbances vs the angle of inclination: $P = 10$.

wide range of angles. However, it should be emphasized that for a vertical orientation the crisis is associated with plane disturbances at all P.

The three-dimensional branch $a = 0$ corresponding to the disturbances of the "z-roll" type may be directly found from the boundary-value problem of the amplitudes of normal disturbances. Assuming in (31) $k_z = 0$ and substituting it into (12)–(14), we arrive at the amplitude problem which does not contain the velocity profile of the main motion:

$$\begin{aligned}
-\lambda v_x &= -p' + (v_x'' - k_y^2 v_x) - T\sin\alpha,\\
-\lambda v_y &= -ik_y p + (v_y'' - k_y^2 v_y),\\
-\lambda v_z &= (v_z'' - k_y^2 v_z) + T\cos\alpha,\\
-\lambda T + G T_0' v_x &= \frac{1}{P}(T'' - k_y^2 T),\\
v_x' + ik_y v_y &= 0,\\
x = \pm 1: v_x = v_y = v_z &= 0, \qquad T = 0.
\end{aligned} \tag{35}$$

As is easily seen, the problem for v_x, v_y, p and T is identical with that for the disturbances of fluid layer at rest with the parabolic temperature distribution.* As is known, this problem has the solution, at which there appear steady disturbances determined from the condition $\lambda = 0$ at the stability boundary. The stability

*The third equation of (35) allows the velocity component of v_z to be found in terms of the temperature disturbance amplitude T. This component differs from zero. Thus, the disturbances of the "z-roll" type correspond to the liquid motion following spiral trajectories.

boundary is found by minimizing $G(k_y)$ and is given by:

$$R_m = G_m P = \frac{R_0}{\sin\alpha} \tag{36}$$

where $R_0 = 584$ is the critical Rayleigh number for a horizontal layer. Formula (36) defines the boundary of the flow stability with respect to the disturbances of the "z-roll" type.

Thus, there exists, in a certain sense, a critical Prandtl number $\tilde{P}$ (approximately estimated $\tilde{P} \approx 0{\cdot}6$). If $P < \tilde{P}$, then for all angles of inclination the crisis of steady motion is of hydrodynamic nature and is caused by plane disturbances. At $P > \tilde{P}$ plane disturbances lead to instability at the angles of inclination to the vertical $\alpha < \alpha_*$ (the phase velocity of these disturbances differs from zero, Fig. 6). At $\alpha > \alpha_*$ the disturbances of the "z-roll" type are responsible for the crisis (the phase velocity of these disturbances is zero). The crisis in the range of the angles is, consequently, of convective nature and is associated with the Rayleigh instability of the stratified fluid.

CONCLUSIONS

Consideration is made of the stability of steady convective motion due to internal heat sources in the plane fluid layer oriented arbitrarily to the vertical. The solution of the amplitude boundary-value problem for plane normal disturbances is obtained by Galerkin's method. The critical Grashof numbers as well as the wave numbers and phase velocities of the critical disturbances are calculated depending on the inclination angle of the layer for three values of the Prandtl number: $P = 0{\cdot}1, 1, 10$. Transformations similar to the Squire one are shown to exist which allow the information on three-dimensional disturbances to be obtained proceeding from the solution of a two-dimensional problem. If $P < 0{\cdot}6$, then throughout the whole range of the angles of inclination the crisis is caused by plane disturbances and is of hydrodynamic nature. If $P > 0{\cdot}6$, then the plane disturbances lead to instability at the angles of inclination to the vertical $\alpha < \alpha_*$ where α_* depends on P. In the range $\alpha > \alpha_*$ the instability is attributable to spiral three-dimensional disturbances of the "z-roll" type. The crisis in this range is conditioned by the Rayleigh instability of the stratified fluid.

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SUR LA STABILITE DU MOUVEMENT CONVECTIF PLAN AVEC DES SOURCES INTERNES DE CHALEUR

Résumé—On résout le problème de la stabilité du mouvement permanent convectif dans une couche de fluide plane et inclinée, avec des sources de chaleur distribuées uniformément. Les nombres de Grashof et les paramètres critiques de perturbation sont déterminés en fonction de l'angle d'inclinaison pour quelques nombres de Prandtl. On établit la relation entre les perturbations bidimensionnelles et tri-dimensionnelles. On montre que, dépendant du nombre de Prandtl et de l'angle d'inclinaison, la crise du mouvement permanent est soit de nature hydrodynamique, soit liée à l'instabilité de Rayleigh du fluide stratifié.

ÜBER DIE STABILITÄT EINER FLÄCHENPARALLELEN KONVEKTIVEN BEWEGUNG BEI INNEREN WÄRMEQUELLEN

Zusammenfassung—Das Problem der Stabilität von stationärer konvektiver Bewegung in der geneigten Fluidschicht mit gleichmäßig verteilten Wärmequellen wird gelöst. Die kritischen Grashof-Zahlen und kritischen Störparameter werden in Abhängigkeit des Neigungswinkels für einige Prandtl-Zahlen bestimmt. Die Beziehungen zwischen 2- und 3-dimensionalen Störungen werden aufgestellt. Es wird gezeigt, daß abhängig von der Prandtl-Zahl und dem Neigungswinkel die Unterbrechung der stetigen Bewegung entweder hydrodynamische Gründe hat oder von der Rayleigh-Instabilität des geschichteten Fluids verursacht wird.

ОБ УСТОЙЧИВОСТИ ПЛОСКОПАРАЛЛЕЛЬНОГО КОНВЕКТИВНОГО ДВИЖЕНИЯ, ВЫЗВАННОГО ВНУТРЕННИМИ ИСТОЧНИКАМИ ТЕПЛА

Аннотация — Решена задача об устойчивости стационарного конвективного движения в плоском наклонном слое жидкости, в которой однородно распределены внутренние источники тепла. Определены критические числа Грасгофа и параметры критических возмущений в зависимости от угла наклона слоя для некоторых значений числа Прандтля. Установлено соотношение между плоскими и пространственными возмущениями. Показано, что в зависимости от числа Прандтля и угла наклона кризис стационарного движения имеет либо гидродинамическую природу, либо связан с рэлеевской неустойчивостью стратифицированной жидкости.